



8.2 – Compositions and Inverse Transformations

Definition: If $T : V \rightarrow W$ is a linear transformation from a vector space V to a vector space W , then T is said to be **one-to-one** if T maps distinct vectors in V into distinct vectors in W .

OR $T : V \rightarrow W$ is one-to-one if and only if for each vector w in the range of T , there is exactly one vector v in V such that $T(v) = w$.

OR $T : V \rightarrow W$ is one-to-one if and only if $T(u) = T(v)$ implies $u = v$.

Definition: If $T : V \rightarrow W$ is a linear transformation from a vector space V to a vector space W , then T is said to be **onto** (or **onto W**) if every vector in W is the image of at least one vector in V .

Examples:

One-to-one and onto:

- Rotation operators on \mathbb{R}^2 $0 \leq \theta < 2\pi$
- $T: P_3 \rightarrow \mathbb{R}^4$, where $T(a+bx+cx^2+dx^3) = (a, b, c, d)$
- $T: M_{22} \rightarrow \mathbb{R}^4$, where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$

One-to-one but not onto:

- $T: P_n \rightarrow P_{n+1}$, where $T(\vec{p}) = T(p(x)) = xp(x)$

Not one-to-one

- $D: C^1(-\infty, \infty) \rightarrow F(-\infty, \infty)$, where

$$D(\vec{f}) = f'(x) \quad \text{Ex: } D(x^2) = D(x^2+1) = 2x$$



Theorem 8.2.1 If $T : V \rightarrow W$ is a linear transformation, then T is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

Pf: (\Rightarrow) By Thm 8.1.1, $T(\vec{0}) = \vec{0}$ because T is linear. If T is one-to-one, then if $T(\vec{v}) = \vec{0}$, then $\vec{v} = \vec{0} \Rightarrow \ker(T) = \{\vec{0}\}$.

(\Leftarrow) Let $\vec{u}, \vec{v} \in V$ such that $\vec{u} \neq \vec{v}$.
Then $\vec{u} - \vec{v} \neq \vec{0} \Rightarrow T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v}) \neq \vec{0}$
 $\Rightarrow T(\vec{u}) \neq T(\vec{v}) \Rightarrow T$ is one-to-one ✓

#4 Determine whether the linear transformation is one-to-one by finding its kernel and then applying Theorem 8.2.1.

a. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, where $T(x, y) = (x - y, y - x, 2x - 2y)$

b. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $T(x, y) = (0, 2x + 3y)$

c. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $T(x, y) = (x + y, x - y)$

a)
$$\begin{array}{l} x - y = 0 \\ y - x = 0 \\ 2x - 2y = 0 \end{array} \rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow x = y$$

$\ker(T) \neq \{\vec{0}\} \Rightarrow T$ is not one-to-one.

b) $\ker(T) = \{(x, y) \mid y = -\frac{2}{3}x\} \neq \{\vec{0}\}$

not one-to-one.

$$c. \quad \begin{array}{l} x+y=0 \\ x-y=0 \end{array} \Rightarrow 2x=0 \Rightarrow x=0 \Rightarrow y=0$$

$$\ker(T) = \{(0, 0)\} \Rightarrow T \text{ is one-to-one.}$$

Theorem 8.2.X If $T : V \rightarrow W$ is a one-to-one linear transformation and $\{v_1, v_2, \dots, v_k\}$ is a linearly independent subset of V , then $\{T(v_1), T(v_2), \dots, T(v_k)\}$ is linearly independent. (one-to-one lin. transf. preserves lin. independence)

pf: Suppose $c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \dots + c_k T(\vec{v}_k) = \vec{0}$.

Because T is linear,

$$T(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k) = \vec{0}$$

Since T is one-to-one, $\sum_{i=1}^k c_i \vec{v}_i = \vec{0}$.

($\ker(T) = \{\vec{0}\}$).

$\Rightarrow c_i = 0, 0 \leq i \leq k$ because

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly independent.

$\Rightarrow \{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_k)\}$ is lin. indep. ✓

↖ $\{T(\vec{v}_i)\}_{i=1}^k$?

T is one-to-one iff $\ker(T) = \{\vec{0}\} \Rightarrow \dim[\ker(T)] = 0$

#5 Determine whether multiplication by A is one-to-one by computing the nullity of A and then applying Theorem 8.2.1

$$\dim[\ker(T_A)] = \text{nullity}(A)$$

a. $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix}$

$$\text{rank}(A) + \text{nullity}(A) = \# \text{ columns}$$

b) $\text{rank}(A) \leq 3 \Rightarrow \text{nullity}(A) \geq 1 \neq 0$
 $\Rightarrow T_A$ is not one-to-one

a) $\text{rref}(A) = \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 1$
 $\text{nullity}(A) = 1 \neq 0$

$\Rightarrow T_A$ is not one-to-one

Theorem 8.2.2 If V and W are finite-dimensional vector spaces with the same dimension, and if $T : V \rightarrow W$ is a linear transformation, then the following statements are equivalent.

- a) T is one-to-one.
- b) $\ker(T) = \{\vec{0}\}$.
- c) T is onto [i.e., $R(T) = W$].

Theorem 8.2.3 If $T_A : R^n \rightarrow R^m$ is a matrix transformation, then

- a) T_A is one-to-one if and only if the columns of A are linearly independent.
- b) T_A is onto if and only if the columns of A span R^m .

#14 Use Theorem 8.2.3 to determine whether multiplication by A is one-to-one, onto, both, or neither. Justify your answer.

a. $A = \begin{bmatrix} 9 & -3 \\ -4 & 2 \\ 1 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -3 & 1 & 1 \\ 6 & -6 & 0 & 2 \\ 9 & -9 & 1 & 3 \end{bmatrix}$

$$\text{c. } A = \begin{bmatrix} 3 & -9 \\ -1 & 3 \end{bmatrix}$$

$$\text{d. } A = \begin{bmatrix} 2 & 3 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

a) 2 col. vectors can't span \mathbb{R}^3

$\vec{c}_1 \neq k\vec{c}_2 \Rightarrow$ cols are indep. $\Rightarrow T_A$ is 1-1

d) $\det(A) = 2 \neq 0 \Rightarrow T_A$ is one-to-one and onto.

Theorem 8.2.4 Equivalent Statements (extends Theorems 4.9.8, 5.1.5 and 6.4.5, the last two of which we have not seen)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n .

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix \mathbf{b} .

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix \mathbf{b} .

g) $\det(A) \neq 0$.

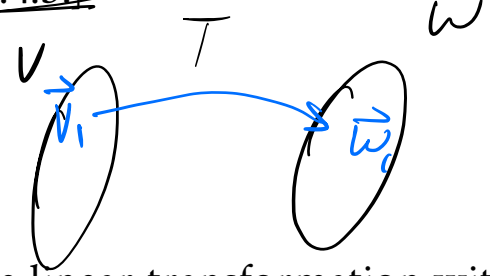
h) The column vectors of A are distinct and linearly independent.

i) The row vectors of A are distinct and linearly independent.

- j) The column vectors of A span R^n .
- k) The row vectors of A span R^n .
- l) The column vectors of A form a basis for R^n .
- m) The row vectors of A form a basis for R^n .
- n) A has rank n .
- o) A has nullity 0.
- p) The orthogonal complement of the null space of A is R^n .
- q) The orthogonal complement of the row space of A is $\{0\}$.
- r) $\lambda = 0$ is not an eigenvalue of A . [This is from Theorem 5.1.5.]
- s) $A^T A$ is invertible. [This is from Theorem 6.4.5.]

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- t) The kernel of T_A is $\{0\}$.
 - u) The range of T_A is R^n .
 - v) T_A is one-to-one.



Definition: If $T : V \rightarrow W$ is a one-to-one linear transformation with range $R(T)$, and if w is any vector in $R(T)$, then there is exactly one vector v in V for which $T(v) = w$. The **inverse of T** maps w back into v and is denoted by T^{-1} . (This is analogous to **inverse** in Section 1.9.)

Definition: (analogous to **composition** in Section 1.9)

If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear transformations, then the **composition** of T_2 with T_1 , denoted by $T_2 \circ T_1$ (which is read " T_2 circle T_1 ") is the function defined by the formula $(T_2 \circ T_1)(u) = T_2(T_1(u))$ where u is a vector in U .

Note that order matters for compositions

Theorem 8.2.5 If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear transformations, then $(T_2 \circ T_1) : U \rightarrow W$ is also a linear transformation.

Theorem 8.2.6 If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are one-to-one linear transformations, then:

- a) $T_2 \circ T_1$ is one-to-one.
- b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

proofs are good exercises

#38 Let $D(\mathbf{f}) = f'(x)$ and $J(\mathbf{f}) = \int_0^x f(t)dt$ be the derivative and integral linear transformations. Find $(J \circ D)(\mathbf{f})$ for

a. $f(x) = x^2 + 3x + 2$

b. $f(x) = \sin x$

$$a) (J \circ D)(x^2 + 3x + 2) = J(2x + 3) = \int_0^x (2t + 3)dt = x^2 + 3x$$

$$P^3 \rightarrow R^4$$

$$M_{22} \rightarrow R^4$$

Key concepts from 8.3:

Definition 1: A linear transformation $T : V \rightarrow W$ that is both one-to-one and onto is said to be an **isomorphism**, and W is said to be **isomorphic** to V .

Theorem 8.3.1 Every real n -dimensional vector space is isomorphic to R^n . [In this sense, every n -dimensional vector space is algebraically equivalent to R^n .]

Theorem 8.3.2 If S is an ordered basis for a vector space V , then the coordinate map $\mathbf{u} \xrightarrow{T} (\mathbf{u})_S$ is an isomorphism between V and R^n .

Theorem 8.3.X A linear transformation $T : V \rightarrow W$ is an isomorphism if and only if whenever $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a basis for V , $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is a basis for W .

Examples of Natural isomorphisms

Between P_{n-1} and R^n : $a_0 + a_1x + \dots + a_{n-1}x^{n-1} \xrightarrow{T} (a_0, a_1, \dots, a_{n-1})$

Between M_{22} and R^4 : $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} (a, b, c, d)$

